Character functions of SU(3)

1988 J. Phys. A: Math. Gen. 21 2651

(http://iopscience.iop.org/0305-4470/21/11/022)

J. Phys. A: Math. Gen. **21** (1988) 2651-2656. Printed in the UK

COMMENT

Character functions of SU(3)

Belal E Baaquie

Department of Physics, National University of Singapore, Kent Ridge, Singapore 0511

Received 4 February 1988

Abstract. We exactly evaluate the character functions of SU(3) from the heat kernel. Our results reproduce the first Weyl formula. We evaluate the Weyl chamber for SU(3) and show its connection to the character functions.

1. Introduction

The heat kernel on SU(N) comprises the matrix elements of $\exp(+\frac{1}{2}\alpha\nabla^2)$ where ∇^2 is the SU(N) Laplace-Beltrami operator [1] and α is a constant. The matrix elements of the pth irreducible representation are eigenfunctions of ∇^2. Let U be an element of SU(N). Then

$$-\nabla^2 \mathscr{D}_{ij}^{(p)}(U) = c(p)\mathscr{D}_{ij}^{(p)}(U) \tag{1.1}$$

where $c(p)$ is the value of the quadratic Casimir operator in the pth irreducible representation.

Let $|U\rangle$ be the coordinate eigenstate of the Hilbert space on SU(N); then the conjugate eigenstate $\langle p, ij|$ satisfies

$$\langle p, ij|U\rangle = \sqrt{\mathrm{d}p}\ \mathscr{D}_{ij}^{(p)}(U) \qquad 1 \leqslant i, j \leqslant \mathrm{d}p \tag{1.2}$$

where dp is the dimension of $\mathscr{D}^{(p)}$. Hence, from (1.1) and (1.2) we have for the SU(N) heat kernel

$$K_N(\alpha) = \langle W|\exp(+\tfrac{1}{2}\alpha\nabla^2)|V\rangle \tag{1.3}$$

$$= \sum_p \exp[-\tfrac{1}{2}\alpha c(p)]\, \mathrm{d}p\, \chi_p(W^+V) \tag{1.4}$$

where

$$\chi_p(U) = \mathrm{Tr}\ \mathscr{D}^{(p)}(U) \tag{1.5}$$

is the character function for the pth irreducible representation, and the sum in (1.4) is over all p. Equation (1.4) is the key equation for identifying the character functions.

In the fundamental representation, let

$$U \equiv W^+V = R\ \mathrm{diag}(\mathrm{e}^{\mathrm{i}A_1}, \ldots, \mathrm{e}^{\mathrm{i}A_N})R^+ \tag{1.6}$$

with R an element of SU(N) and

$$\sum_{I=1}^{N} A_I = 0. \tag{1.7}$$

0305-4470/88/112651+06\$02.50

For SU(3), we have for (1.6)

$$U = R\,\mathrm{diag}(e^{iA}, e^{iB}, e^{-i(A+B)})R^{+}. \tag{1.8}$$

The irreducible representations of SU(3) are labelled by two positive integers (p, q) and we choose the convention that (1, 1) is the one-dimensional (trivial) representation, $\underset{\sim}{3}$ is (2, 1) and $\underset{\sim}{3}^*$ is (1, 2). It is well known that for SU(3)

$$d_{p,q} = \tfrac{1}{2}pq(p+q) \qquad p, q = 1, 2, \ldots, \infty \tag{1.9a}$$

$$c(p, q) = \tfrac{1}{3}(p^2+q^2+pq) - 1 \tag{1.9b}$$

and

$$\chi^*_{p,q}(A, B) = \chi_{q,p}(A, B) \tag{1.9c}$$

$$\chi_{p,q}(A, B) = -\chi_{p,q}(B, A). \tag{1.9d}$$

Hence, from (1.4), (1.8) and (1.9), we have up to constants

$$K_3 = \sum_{p,q=1}^{\infty} \tfrac{1}{2}pq(p+q)\tfrac{1}{2}[\chi_{p,q}(A, B) + \chi^*_{p,q}(A, B)] \exp[-\tfrac{1}{2}\alpha\tfrac{1}{3}(p^2+q^2+pq)]. \tag{1.10}$$

2. Heat kernel for SU(3)

Using differential [2] or functional integral methods [3], the heat kernel can be evaluated explicitly. The result for SU(N) is

$$K_N(\alpha) = \prod_{I=1}^{N} \sum_{l_I=-\infty}^{+\infty} \left(\prod_{I<J} \frac{A_I - A_J + 2\pi l_I - 2\pi l_J}{\sin\frac{1}{2}(A_I - A_J + 2\pi l_I - 2\pi l_J)} \right) \delta\left(\sum_I l_I\right) \times \exp\left(-\frac{1}{2\alpha} \sum_I (A_I + 2\pi l_I)^2 \right). \tag{2.1}$$

For SU(3), we have from (1.8), using $A_1 = A$ and $A_2 = B$, the following:

$$K_3(\alpha) = \frac{1}{s(A, B)} \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} [A(l) - B(m)][A(l) + 2B(m)][2A(l) + B(m)] \times \exp\{-(1/\alpha)[A^2(l) + B^2(m) + A(l)B(m)]\} \tag{2.2}$$

where

$$s(A, B) = 8 \sin\tfrac{1}{2}(A-B) \sin\tfrac{1}{2}(2A+B) \sin\tfrac{1}{2}(A+2B) \tag{2.3}$$

and

$$A(l) = A + 2\pi l \qquad B(m) = B + 2\pi m. \tag{2.4}$$

Comparing (2.2) with (1.10) will yield the character functions of SU(3). The main obstacle in converting (2.2) into (1.10) is that (2.2) is an expansion of K_3 in powers of $\exp(-1/\alpha)$ whereas (1.10) is in powers of $\exp(-\alpha)$. The other difference is that, in (1.10), we sum only over integers p, q such that $p, q > 0$, whereas in (2.2) the sum in l, m is over all integers.

We first transform equation (2.2) by applying the Poisson summation formula, namely

$$\sum_{l=-\infty}^{+\infty} f_l = \sum_{p=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \exp(2\pi i p x) f_x \tag{2.5}$$

where f_x is any 'reasonable' continuation of f_l to non-integral values of l. Hence, from (2.2) and (2.5) we have

$$K_3(\alpha)=\frac{1}{s(A,B)}\sum_{p,q=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}pA}\,\mathrm{e}^{\mathrm{i}qB}\,K_3'(p,q) \tag{2.6}$$

where

$$K_3'(p,q)=\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{d}y$$

$$\times\exp(\mathrm{i}px-\mathrm{i}qy)(x-y)(x+2y)(2x+y)\exp[-(2/\alpha)(x^2+y^2+xy)] \tag{2.7}$$

$$=\text{constant}\times\mathrm{i}pq(p+q)\exp[-\tfrac{1}{2}\alpha\cdot\tfrac{1}{3}(p^2+q^2+pq)]. \tag{2.8}$$

Therefore, from (2.6) and (2.8), after some simplifications

$$K_3(\alpha)=\sum_{p,q=-\infty}^{+\infty} d_{p,q}\exp[-\tfrac{1}{2}\alpha c(p,q)]\Omega_{p,q}(A,B) \tag{2.9a}$$

with

$$\Omega_{p,q}(A,B)=\frac{1}{s(A,B)}[\sin(pA-qB)+\sin(qA-pB)]. \tag{2.9b}$$

Comparing (2.9a) with (1.10) we note that the p, q summations in (2.9) are over four quadrants in the pq plane, whereas in (1.10) it is only over the first quadrant. We hence have to restrict the sum in (2.9a) to the first quadrant, and this will be possible due to certain symmetries of $d_{p,q}$ and $c(p,q)$ (see figure 1).

We have

$$d_{p,q}=d_{q,p} \tag{2.10a}$$

$$=-d_{-p,-q} \tag{2.10b}$$

and

$$c(p,q)=c(q,p) \tag{2.11a}$$

$$=c(-p,-q). \tag{2.11b}$$

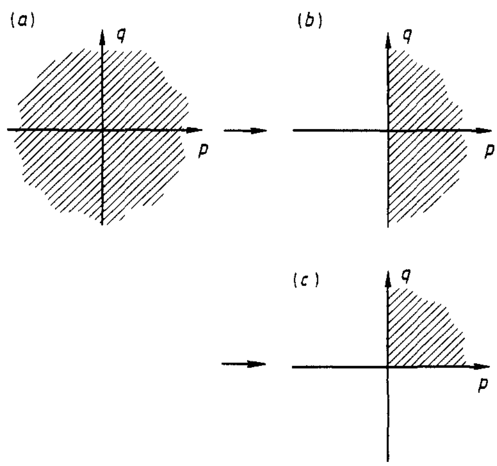

Figure 1. Reduction of summation from four quadrants in (p,q) to the first quadrant. Shaded portions are summed over. For the significance of (a)–(c) see the text.

Using (2.10b) and (2.11b) we reduce (2.9) to a sum over the first and fourth quadrant, i.e. (figure 1(b))

$$K_3(\alpha)=\sum_{p=1}^{\infty}\left(\sum_{q=1}^{\infty}+\sum_{q=-\infty}^{-1}\right)d_{p,q}\Omega_{p,q}\exp[-\tfrac{1}{2}\alpha c(p,q)] \tag{2.12}$$

$$\equiv K_3^{(1)}+K_3^{(2)}. \tag{2.13}$$

The first sum in (2.12) is over the first quadrant in (p, q), but the second sum is over the fourth quadrant and has to be further reduced.

We have

$$K_3^{(2)}=\sum_{p=1}^{\infty}\sum_{q=-\infty}^{-1}d_{p,q}\exp[-\tfrac{1}{2}\alpha c(p,q)]\Omega_{p,q} \tag{2.14}$$

$$=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}d_{p,-q}\exp[-\tfrac{1}{2}\alpha c(p,-q)]\Omega_{p,-q}. \tag{2.15}$$

Using the fact that

$$c(p,-q)=\begin{cases}c(p-q,q) & p>q\\ c(p,q-p) & p<q\end{cases} \tag{2.16}$$

we have

$$K_3^{(2)}=\sum_{p=1}^{\infty}\left(\sum_{q=1}^{p-1}+\sum_{q=p+1}^{\infty}\right)d_{p,-q}\exp[-\tfrac{1}{2}\alpha c(p,-q)]\Omega_{p,-q} \tag{2.17}$$

$$=\sum_{q=1}^{\infty}\sum_{p=q+1}^{\infty}d_{p,-q}\exp[-\tfrac{1}{2}\alpha c(p-q,q)]\Omega_{p,-q}$$

$$+\sum_{p=1}^{\infty}\sum_{q=p+1}^{\infty}d_{p,-q}\exp[-\tfrac{1}{2}\alpha c(p,q-p)]\Omega_{p,-q}. \tag{2.18}$$

Rearranging the summations in (2.18) we have (figure 1(c))

$$K_3^{(2)}=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}d_{p,q}\exp[-\tfrac{1}{2}\alpha c(p,q)][-\Omega_{p+q,-q}+\Omega_{p,-(p+q)}]. \tag{2.19}$$

Hence we have, from (2.13), (2.14) and (2.19)

$$K_3(\alpha)=\sum_{p,q=1}^{\infty}d_{p,q}\exp[-\tfrac{1}{2}\alpha c(p,q)][\Omega_{p,q}-\Omega_{p+q,-q}+\Omega_{p,-(p+q)}]. \tag{2.20}$$

3. The character functions

By comparing (1.10) and (2.20) we have for SU(3) character functions

$$\chi_{p,q}=-\frac{\mathrm{i}}{s(A,B)}\{\exp(\mathrm{i}pA-\mathrm{i}qB)-\exp(-\mathrm{i}qA+\mathrm{i}pB)$$

$$+\exp[-\mathrm{i}p(A+B)](\exp(-\mathrm{i}qA)-\exp(-\mathrm{i}qB))$$

$$+\exp[\mathrm{i}q(A+B)](\exp(\mathrm{i}pB)-\exp(\mathrm{i}pA)]\} \tag{3.1}$$

where, from (2.3)

$$s(A,B)=8\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(A+2B)\sin\tfrac{1}{2}(2A+B). \tag{3.2}$$

Note that $\chi_{p,q}$ satisfies (1.9*c*) and (1.9*d*); for the $\underset{\sim}{3}$ (fundamental) representation, as is expected from (1.8), we have

$$\chi_{2,1}(A, B) = e^{iA} + e^{iB} + e^{-i(A+B)}. \tag{3.3}$$

We also have the non-trivial identity

$$\chi_{p,q}(0, 0) = \tfrac{1}{2}pq(p+q) \tag{3.4a}$$

$$= d_{p,q}. \tag{3.4b}$$

The expression for $\chi_{p,q}$ in (3.1) is a ratio of two determinants and is the first Weyl formula for the character functions [4]. Weyl's derivation uses the integral properties of the group. A purely algebraic derivation of $\chi_{p,q}$ can be given using the Lie algebra [5]. The derivation given here is independent of these two derivations.

It is well known that the invariant measure on SU(3) for class functions is given by [4]

$$dU = s^2(A, B)\, dA\, dB \qquad -\pi \leqslant A, B \leqslant \pi. \tag{3.5}$$

where U is an element of SU(3). With this measure we have the expected orthonormality theorem given by [4]

$$\int_{-\pi}^{+\pi} \frac{dA\, dB}{(2\pi)^2} s^2(A, B)\chi^*_{(p,q)}(A, B)\chi_{(p',q')}(A, B) = \delta_{pp'}\delta_{pp'}. \tag{3.6}$$

Further simplification of (3.6) can be made. Since $\chi_{p,q}$ depends only on the trace of U, it is invariant under the Weyl group W; this discrete and finite subgroup of SU(N) consists of reflections of root vectors in the root space, and for SU(3) consists of six elements. For an element $T \in$ W, we have

$$W: U \to T^{-1}UT. \tag{3.7}$$

The Weyl chamber in the SU(3) group space consists of points A, B such that

$$s(A, B) \geqslant 0. \tag{3.8}$$

The total space U(1)⊗U(1) spanned by A and B is split into six disjoint chambers, each a reflection of the compact domain Γ defined from (3.8) by (figure 2)

$$\Gamma: A \geqslant B \qquad 0 \leqslant A + 2B \leqslant 2\pi \qquad 0 \leqslant 2A + B \leqslant 2\pi. \tag{3.9}$$

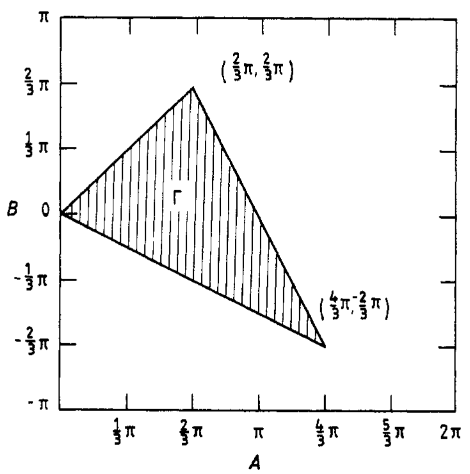

Figure 2. Domain of the Weyl chamber Γ in the toral space of (A, B) signified by the shaded portion.

The orbit of the Weyl chamber Γ under the action of the Weyl group is the total space U(1)⊗U(1) ([1], p 316). We hence have, using the Weyl group

$$\frac{1}{A_\Gamma}\int_\Gamma \mathrm{d}A\,\mathrm{d}B\, s^2(A, B)\chi^*_{(p,q)}(A, B)\chi_{(p,q)}(A, B) = \delta_{pp'}\delta_{qq'} \qquad (3.10)$$

$$A_\Gamma = \text{area of } \Gamma = \tfrac{4}{6}\pi^2. \qquad (3.11)$$

The form (3.10) with integration over Γ which is one sixth the area of the torus is more suitable for numerical calculations. Also, to prove (3.10) directly without using the Weyl group is cumbersome and complicated.

4. Conclusions

We derived the SU(3) character functions from the SU(3) heat kernel, a derivation independent from Weyl's classic formulae. Certain non-trivial symmetries of the dimensionality and Casimir function were central to the derivation. It is not possible to derive the SU(N) character functions from the SU(N) heat kernel, since we need $N-1$ algebraic invariants for it, whereas the heat kernel uses only two of these.

Acknowledgment

I would like to thank Professor C K Chew for helpful discussions.

References

[1] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)
[2] Menotti P and Onofri E 1981 *Nucl. Phys.* B **190** 288
[3] Baaquie B E 1985 *Phys. Rev.* D **32** 1007
[4] Weyl H 1935 *Classical Theory of Groups* (Princeton, NJ: Princeton University Press)
[5] Jacobson N 1979 *Lie Algebras* (New York: Dover)